RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 49, NUMBER 5 MAY 1994

Order in two-dimensional oscillator lattices

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(Received 23 November 1993)

We study phase and frequency ordering in a square lattice of limit-cycle oscillators with randomly distributed natural frequencies. We measure the cluster distribution in lattices of increasing size and we find that macroscopic frequency ordering is possible in two dimensions in contrast with previous results.

PACS number(s): 05.40.+j

In the past few years several authors addressed the issue of macroscopic synchronization in populations of limit cycle oscillators. Sakaguchi, Shinomoto, and Kuramoto (SSK) [1], Daido [2], and Strogatz and Mirollo (SM) [3] have studied this problem with a simple model based on a phase description of individual oscillators:

$$\dot{\theta}_j = \Omega_j + \sum_{\langle i,j \rangle} h_{ij}(\theta_i - \theta_j, K). \tag{1}$$

Here θ_j and Ω_j are the phase and natural frequency of the jth oscillator. h_{ij} is an odd coupling function periodic in its first argument, and K is the intensity of the coupling. The natural frequencies are assumed to be independent random variables obeying some distribution function $f(\Omega)$. The summation is over the set of nearest neighbors of oscillator j. As a result of the coupling each oscillator will have an asymptotic frequency defined as

$$\omega_j = \lim_{T\to\infty} \frac{1}{T}[\theta_j(t+T) - \theta_j(t)], \tag{2}$$

which is expected to be independent of t.

Two oscillators are said to be synchronized, or entrained, when their asymptotic frequencies are the same, which implies that their phase difference is independent of time. For some $f(\Omega)$, it is possible to observe the formation of clusters of entrained oscillators which, depending on the value of K, will comprise the whole finite system. The question is, then, is it possible to observe complete synchronization in an infinite system for a finite value of K? This is the key question to be answered if one wants to interpret the onset of synchronization as a nonequilibrium phase transition.

Ordering, in the sense of synchronization, will depend on the coupling intensity, on the natural frequency distribution, and on the lattice dimension. SSK, Daido, and SM have considered these effects on the existence of a critical dimension d_c for macroscopic synchronization. We summarize their results below.

SSK measured the macroscopic synchronization by means of an *order parameter* defined as

$$r = \lim_{N\to\infty} \frac{N_s}{N}, \tag{3}$$

where N_s is number of oscillators in the largest synchronized cluster. With this definition, the question above can be rephrased as "Is it possible to have a nonzero r for finite K?" They also considered the existence of a phase order characterized by another *order parameter*:

$$\sigma = \frac{1}{N}\sum_{j=1}^{N} \exp(i\theta_j). \tag{4}$$

$\sigma = 0$ means that the phases are randomly distributed. By heuristic arguments and computer simulations of a hypercubic lattice of $N = L^d$ oscillators with Gaussian distributed frequencies they found that phase order is possible only for $d > 4$, and frequency order appears for $d > 2$.

Daido has studied Eq. (1) using a renormalization-group analysis. He found that frequency order is possible only when $d > \frac{\alpha}{\alpha-1}$, where α is defined as $f(\Omega) \propto |\Omega|^{-\alpha-1}$, ($|\Omega| \gg 1$). For a Gaussian distribution of native frequencies, $\alpha = 2$, so frequency order is possible for $d > 2$.

Finally, SM found that the probability of synchronization in a hypercubic lattice tends to zero exponentially fast as the number of oscillators grows without bound, independently of the lattice dimension.

Since results of simulations and analytical calculation do not agree, we decided to study the problem of macroscopic synchronization by means of a cell dynamical system (CDS) [4,5] which would give us the complete oscillatory solution. Also we decided to improve SSK's results, using larger lattices and better statistics. To our surprise the results for the two-dimensional lattice were quite different from SSK's, as will be shown below. We did not consider three-dimensional systems because we found out that (1) generates an extremely inefficient code for simulations. The simulations with the CDS model are still in progress and are planned to be published elsewhere.

Here we followed SSK closely and considered the coupling between nearest neighbors of the form

$$h_{ij}(\theta_i - \theta_j, K) = K \sin(\theta_i - \theta_j), \quad K > 0 \tag{5}$$

and natural frequencies obeying a Gaussian distribution with unit variance. Equation (1) was then integrated using an Euler scheme:

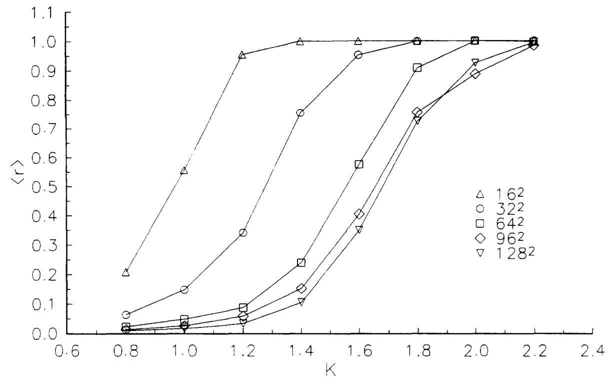

FIG. 1. Frequency order parameter as a function of the coupling intensity for two-dimensional systems. As the system size increases, the curves converge suggesting the existence of order as $N \to \infty$.

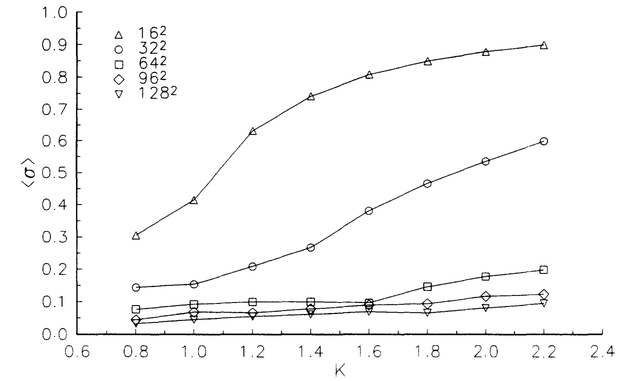

FIG. 2. Phase order parameter as a function of the coupling intensity for two-dimensional systems. As the system size increases $\langle\sigma\rangle \to 0$ indicating that phase order is absent.

$$\theta_i(t+\Delta t) - \theta_i(t) = \Delta t\left(\Omega_i + K\sum_{\langle ij\rangle}\sin[\theta_i(t) - \theta_j(t)]\right) \quad (6)$$

with $\Delta t = 0.1$. We calculated r and σ in lattices of different sizes, for a range of values of K after 9000 iterations of (6). The initial phases were chosen to be zero. The 3000 initial iterations were discarded in the calculation of the asymptotic frequencies. We considered that a pair of nearest neighbors had the same frequency whenever their frequency difference was less than π/T. The cluster distribution was then calculated with a routine based on an algorithm used in percolation problems [6]. Figure 1 shows the plot of $\langle r\rangle$ ($\langle r\rangle$ is the average of r over 20 samples) as a function of K for systems of different sizes. The curves present a phase-transition-like behavior that remains as the system size is increased. Actu-

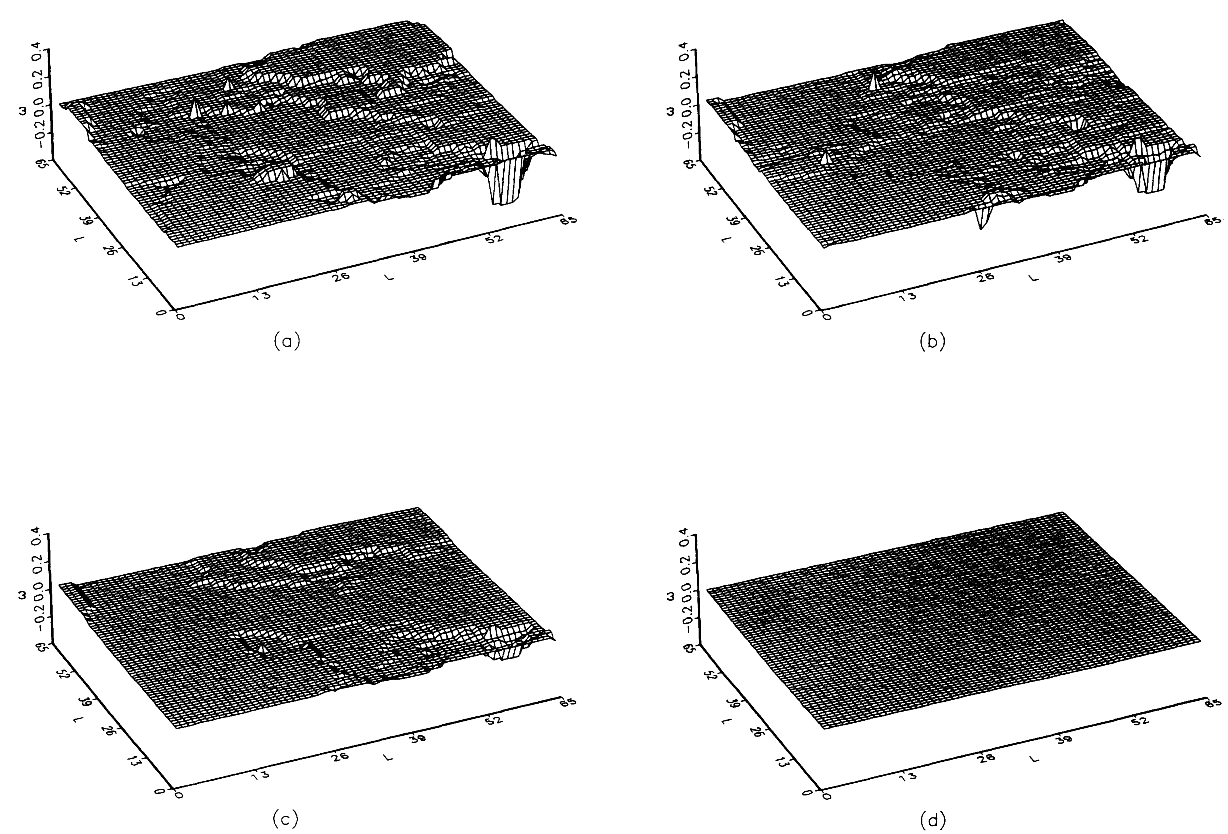

FIG. 3. Formation of clusters of oscillator with the same asymptotic frequency in a 64×64 system for (a) K=1.6, (b) K=1.8, (c) K=2.0, and (d) K=2.2.

ally, the curves converge to a single curve in the limit of large N. This same behavior was found by SSK in three-dimensional systems only [7].

In Fig. 2 we plot the phase order parameter, $\langle\sigma\rangle$ versus K (the average is also over 20 samples). In this case as N increases there is no evidence that σ will be nonzero. We can say that the two-dimensional case presents frequency order without phase order.

Figure 3 illustrates the formation of a macroscopic cluster as K increases; the flat regions correspond to clusters of oscillators with the same frequency for a lattice with $N = 64^2$.

Our conclusion is that frequency order is possible in two-dimensional lattices, in disagreement with SSK, Daido, and SM. This result shows that the existence of a critical dimension for macroscopic synchronization is not clear. We expect to be able to obtain more conclusive results with the study of the complete oscillatory solution.

One of us (M.S.O.M.) was supported by CNPq (Brazil). For the simulations we used the Cray Y-MP at the National Center for Supercomputing at the Federal University of Rio Grande do Sul (Brazil).

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